PMM U.S.S.R.,Vol.48,No.6,pp.750-757,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

SPATIAL CONTACT PROBLEMS FOR ROUGH ELASTIC BODIES UNDER ELASTOPLASTIC DEFORMATIONS OF THE UNEVENNESS*

B.A. GALANOV

On the basis of /1, 2/, a model is constructed for the contact between a rigid stamp and a rough body taking elastoplastic deformations of the unevenness into account. The contact model for rough bodies with elastic deformations of the unevenness is a special case. A classical approach utilizing boundary integral equations is applied in the mathematical formulation of the contact problem. Under quite general assumptions (for instance, the multiconnectedness of the contact domain desired), the uniqueness and existence of the solution are investigated. A method is developed to determine the contact pressure, the closure of the bodies, and also the contact area which consists of two parts in the general case, a zone of elastoplastic deformation of the unevenness and a zone of their elastic deformation. The efficiency of the method is shown in examples of new contact problems. The solution is represented in a convenient form for analysing the influence of the roughness. This is of considerable value for material testing by a contact method. A fairly complete survey of research on contact problems for rough bodies can be found in /1—4/.

1. Formulation of the problem. We examine the contact problem for a rough elastic half-space $z \geqslant 0$. A rigid stamp bounded by the surface $z = -\beta f(x, y) < 0$, $\beta = \text{const} > 0$. is impressed into the half-space by a force P. The stamp occupies the domain $z \leqslant -\beta f(x, y)$. It is assumed that $f(x, y) \in C(E_2)$ (E_2 is the $z = 0$ plane).

We shall assume the normal displacements $w(M)$ of the half-space boundary at the point $M(x, y)$ to have the form $w(M) = w_1(M) + w_2(M)$ and no friction between the stamp and half-space. Here $w_1(M)$ are displacements because of elastoplastic deformations of the unevenness (a rough layer), and $w_2(M)$ are elastic displacements of the half-space.

According to the results obtained in /1, 2/, the displacements $w_1(M)$ are a function of the contact pressure $p(M)$, i.e., $w_1(M) = \Phi(p(M))$. The specific form of the functional connection $w_1 = \Phi(p)$ will be examined below.

The displacements $w_2(M)$ are defined thus /5/

$$w_2(M) = \theta \int_S K(M, N)\, p(N)\, dS_N$$

$$K(M, N) = R_{MN}^{-1} = [(x-\xi)^2 + (y-\eta)^2]^{-1/2}, \quad \theta = \frac{1-\sigma^2}{\pi E}$$

where $S \subset E_2$ is the contact area, R_{MN} is the spacing between the points $M(x, y)$ and $N(\xi, \eta)$, and E and σ are Young's modulus and Poisson's ratio.

Let $h > 0$ be the settling of the stamp, and $g(M) = h - \beta f(M)$. Then the geometric condition of stamp contact with the half-space $w_1(M) + w_2(M) = g(M)$ has the form

$$\Phi(p(M)) + \theta \int_S K(M, N)\, p(N)\, dS_N = g(M); \quad M, N \in S$$

Therefore, the contact problem reduces to finding the quantities $p(M)$, S, h from the system (L is the boundary of S)

$$\Phi(p(M)) + \theta \int_S K(M, N)\, p(N)\, dS_N = g(M) \tag{1.1}$$

$$\int_S p(N)\, dS_N = P; \quad p(M) \geqslant 0, \quad p(L) = 0, \quad M \in S$$

$$\theta \int_S K(M, N)\, p(N)\, dS_N > g(M); \quad p(M) = 0, \quad M \in (E_2 \setminus S)$$

For elastic strains of the unevenness the function $\Phi(p)$ has the form /1—4/

$$w_1 = \Phi(p) \equiv A p^\alpha; \quad A = \text{const} > 0, \quad 0 < \alpha = \text{const} \leqslant 1 \tag{1.2}$$

If $p \geqslant [p_c] = \text{const} > 0$, then the unevennesses are deformed elastoplastically and the

*Prikl.Matem.Mekhan.,48,6,1020-1029,1984 function $w_1 = \Phi(p)$ is represented implicitly by the formulas (/2/, p.86)

$$pD\,(c\sigma_T D)^{-2\nu-1}\left(\frac{\Delta}{5{,}4}\right)^{\nu} = \frac{\sqrt{5{,}4}\,K_\nu}{2\sqrt{\pi}}\, I_\psi\left(\frac{5}{2},\ \nu-1\right)\psi^{-\nu-1/2} + \left(\frac{1-\psi}{\psi}\right)^{\nu-1}\left(\frac{1-\psi}{\psi}+\nu\right) \equiv \varphi(\psi);\quad \psi = R_{\max}\frac{\delta_k}{w_1},\quad D = \pi\theta \tag{1.3}$$

Here I_ψ is the ratio of the incomplete beta functions, σ_T is the yield point, and ν is the exponent of the reference curve of the surface roughness profile (see /2/ for the remaining notation in (1.3)).

Formulas (1.2) and (1.3) show that the relation $w_1 = \Phi(p)$ is determined by both the geometric parameters of the rough layer and by the elastoplastic properties of the half-space material. The constant $[p_c]$ (see /2/) indicates the contact pressure level above which a certain fraction of the unevenness starts to be deformed plastically, i.e., determines the domain of elastoplastic deformation of the unevennesses $\omega = \{M : p(M) \geqslant [p_c]\};\ \omega \subset S$.

The existence of a strictly monotonic continuous function $w_1 = \Phi(p)$ for $p \geqslant 0$ follows from the strict monotonicity of the continuous function $\varphi(\psi)$.

We shall later consider the function $\Phi(p)$ continued to values $p < 0$ by means of the formula $w_1 = A \operatorname{sign} p\, |p|^{\alpha}$ and shall denote the inverse function for Φ by H. The function $p = H(w_1)$ is defined for all w_1, is continuous, grows strictly, $0 = H(0)$, and a constant c_* exists such that $|H(w_1)| \leqslant c_* |w_1|^{1/\alpha}$.

In addition, we assume with respect to the function $f(M)$ that for any $h < \infty$ a bounded domain $\Omega_h = \{M : g(M) > 0\}$ exists and $g(M) \leqslant 0$ for $M \notin \Omega_h$ (the domain Ω_h can be multi-connected). Obviously $\Omega_{h_1} \subset \Omega_{h_2}$ for $h_2 > h_1$.

Proposition 1. If $\{p, S, h\}$ is a solution of system (1.1), then $S \subset \overline{\Omega}_h$ and $p(M) \in C(E_2)$.

The first inclusion $S \subset \overline{\Omega}_h$ is evident. The second inclusion is proved by reductio ad absurdum.

This is an alternative: 1) $p(M),\ M \in S$ is a discontinuous bounded measurable function; 2) $p(M),\ M \in S$ is an unbounded measurable function. For the former possibility the sum $\Phi(p(M)) + w_2(M)$ is a discontinuous bounded function (since $w_2(M)$ is continuous /6/ and $\Phi(p)$ is a strictly monotonic continuous function), which contradicts the continuity of $g(M)$. For the latter possibility the sum $\Phi(p(M)) + w_2(M)$ of non-negative functions, at least one of which is unbounded, is unbounded. This again contradicts the continuity of $g(M)$. The contradictions obtained prove Proposition 1.

Let us introduce the positively homogeneous bounded operator Q which sets $u(M),\ M \in \Omega$ in correspondence with the function $u^+(M),\ M \in \Omega$ by means of the rule

$$u^+(M) = Q(u(M)) = \max\{u(M),\ 0\}$$

For $u \in C(\Omega)$ and $u' \in C(\Omega)$ we have

$$\|u^+\| \leqslant \|u\|,\quad |Qu - Qu'| \leqslant |u - u'| \leqslant \|u - u'\|$$

With respect to the unknown function $u(M)$, we examine the Hammerstein equation in $C(\Omega)$:

$$u(M) - \theta\int_\Omega K(M, N)\, Q(H(g(N) - u(N)))\, dS_N = 0;\quad M, N \in \Omega \tag{1.4}$$

where Ω is an arbitrary bounded domain containing the closure of Ω_h.

If $u(M)$ is a solution of (1.4), then $p(M) = Q(H(g(M) - u(M)))$, $S = \{M : g(M) \geqslant u(M)\}$ and h is a solution of system (1.1) for $P = \int_\Omega Q(H(g(N) - u(N)))\, dS_N$, where $S \neq \varnothing$ for $\Omega_h \neq \varnothing$. Conversely, if $\{p, S, h\}$ is a solution of system (1.1), then the function

$$u(M) = \begin{cases} g(M) - \Phi(p(M)), & M \in S \\ \theta\int_S K(M, N)\, p(N)\, dS_N, & M \in (\Omega \setminus S) \end{cases}$$

is a solution of (1.4).

We later set $\Omega = \{M : \beta f(M) \leqslant h_0\}$ in (1.4) for all $h \in [0, h_0]$.

To reduce the writing, we represent (1.4) in the operator form

$$u - \theta KQH(g - u) = 0;\ u \in C,\ g \in C \tag{1.5}$$

and use the notation

$$g_0(M) = h_0 - \beta f(M);\ \theta_1 = r_0 c_*^{-1} \|K\|^{-1} (\|g_0\| + r_0)^{-1/\alpha}$$

$$P_0 = \int_\Omega Q(H(g_0(N) - u_0(N)))\, dS_N;\quad U = \{u : \|u\| \leqslant r_0\}$$

Here KQH is a completely continuous operator in $C(\Omega)$, and u_0 is the solution of (1.5)

for $h = h_0$ (h_0 is the settling of the stamp under the force P_0).
The sufficient conditions for problem (1.1) to be solvable are set by.

Proposition 2. If $\theta \leqslant \theta_1$, then system (1.1) has a unique solution for $P \in [0, P_0]$:

$$p(M) = Q(H(g(M) - u(M))) \in C(\Omega)$$
$$S = \{M : g(M) \geqslant u(M)\} \subset \Omega;\ h \in [0, h_0]$$

($u(M) \in C(\Omega)$ is a solution of (1.4)) that exists for all $h \in [0, h_0]$ ($p(M) = 0$ for $M \notin \Omega$). Here $h = h(P)$, $P \in [0, P_0]$ is a continuous, strictly increasing function while the domain S can be multiconnected.

Proof. We consider the sphere $U = \{u : \| u \| \leqslant r_0\}$ and the operator $Gu \equiv \theta KQH(g - u)$, $u \in C(\Omega)$. For $h = h_0$ and $u \in U$ we have

$$\| Gu \| \leqslant \theta c_* \| K \| \| g_0 - u \|^{1/\alpha} \leqslant \theta c_* \| K \| (\| g_0 \| + r_0)^{1/\alpha} \leqslant \theta_1 c_* \| K \| (\| g_0 \| + r_0)^{1/\alpha} = r_0$$

i.e., the operator G converts the sphere U into itself and has a fixed point u_0 there (according to the Schauder theorem). The following solution of system (1.1) corresponds to this point for $P = P_0$:

$$p(M) = Q(H(g_0(M) - u_0(M))),\ S = \{M : g_0(M) \geqslant u_0(M)\},\ h = h_0$$

Since $\| g \| \leqslant \| g_0 \|$ and $\Omega_h \subset \Omega$ for $h \in [0, h_0]$, then the existence of the solution $u(M)$ of (1.5) is proved analogously for all $h \in [0, h_0]$, and therefore, the existence also of the function

$$P(h) = \int_\Omega Q(H(g(M) - u(M)))\, dS_M;\quad h \in [0, h_0]$$

Let u_1, u_2 be solutions of (1.5) corresponding to the values $h = h_1 \leqslant h_0$ and $h = h_2 \leqslant h_0$. We introduce the notation

$$(a, b) = \int_\Omega a(N)\, b(N)\, dS_N,\quad v = g - u$$
$$g_1 = h_1 - \beta f,\ g_2 = h_2 - \beta f,\ v_1 = g_1 - u_1$$
$$v_2 = g_2 - u_2,\ \varepsilon = v_2 - v_1,\ d = QHv_2 - QHv_1$$

and we write (1.5) in the form

$$v + \theta KQHv = g;\ v \in C,\ g \in C \tag{1.6}$$

We can obtain from (1.6)

$$\varepsilon + \theta Kd = h_2 - h_1 \tag{1.7}$$
$$(\varepsilon, d) + \theta(Kd, d) = (h_2 - h_1)(P(h_2) - P(h_1))$$
$$(\varepsilon, \varepsilon) + \theta(Kd, \varepsilon) = (h_2 - h_1, \varepsilon)$$

Because of the properties Q and the strict growth H the following inequalities hold:

$$d(M) \geqslant 0 \text{ when } \varepsilon(M) > 0;\ d(M) \leqslant 0 \text{ when } \varepsilon(M) < 0 \tag{1.8}$$

Hence (since K is a strictly prositive operator), the second equation in (1.7) is possible for $h_1 = h_2$ only for $d = 0$. This means that $\varepsilon \equiv 0$ (the corollary of the first equality in (1.7)). In other words, (1.6) and (1.5) can have just one solution. Consequently, system (1.1) can also have just one solution $\{p, S, h\}$.

Now, let $h_1 \neq h_2$. Since $d \not\equiv 0$ here, then strict growth of the functions $P = P(h)$ and $h = h(P)$ follow from the second equation in (1.7) and (1.8).

Let the quantity h_1 be fixed and $h_2 \to h_1$. Then (since $(Kd, \varepsilon)^2 \leqslant (Kd, d)(K\varepsilon, \varepsilon)$, $(Kd, d) \to 0$, the functions ε are bounded in the set for all $h_2 \leqslant h_0$), it follows from the last equation in (1.7) that $(\varepsilon, \varepsilon) \to 0$. Hence, and from the continuity of the operator QH we obtain

$$P(h_2) - P(h_1) = \int_\Omega (QH(v_2(N)) - QH(v_1(N)))\, dS_N \to 0,\quad h_2 \to h_1$$

i.e., the function $P = P(h)$ is continuous for $h \in [0, h_0]$. Consequently (taking account of the strict increase of $P(h)$), we conclude that the function $h = h(P)$ is continuous and increases strictly. The proof of Proposition 2 is thereby completed.

Therefore, the problem of solving system (1.1) is equivalent to the problem of seeking the pair $\{u(M), h\}$ from a system consisting of (1.5) and the equation

$$P = \int_\Omega Q(H(g(N) - u(N)))\, dS_N$$

Different approximate methods, in which the most awkward element is the process of solving (1.5), can be used to solve the system mentioned.

For $| w_1 | \leqslant \| g_0 \| + r_0$ let the function $H(w_1)$ satisfy the Lipschitz condition with constant L, $\theta_2 = L^{-1} \| K \|^{-1}$, $\theta < \min\{\theta_1, \theta_2\}$, $h \in [0, h_0]$. Then the method of successive approximations

$$u_{n+1} = \theta KQH(g - u_n),\ u_0 = 0;\ n = 0, 1, 2, \ldots$$

reduces to solving (1.5). Other approximate method /7, 8/ can also be used to solve (1.5).

All the constructions made above can be performed in broader spaces can $C(\Omega)$. Moreover, the method of investigation considered possesses considerable generality and can be extended to contact problems taking account of the friction forces and moments acting on the stamp.

In addition to the constraints on $f(M)$ assumed above, we shall later assume that $f(M)$ is a positively homogeneous function of degree M.

We make a change of variables in (1.1)

$$x = \varkappa x_*;\ y = \varkappa y_*;\ \varkappa = (h/\beta)^{1/m};\ S = H_0^\varkappa (S_*)$$
$$p(M) = \theta^{-1}(h^{m-1}\beta)^{1/m} F(q(M_*));\quad \mu = A\theta^{-\alpha}(\beta^\alpha h^{(m-1)\alpha-m})^{1/m} \geqslant 0$$

Here

$$F(q) = \begin{cases} \operatorname{sign} q\,|q|^{1/\alpha}, & q \leqslant [q] \\ 2K_\nu^{-1}(\pi/5{,}4)^{1/2}[q]^{1/\alpha}\varphi\left(\frac{[q]}{q}\right), & q > [q] \end{cases}$$

$H_0^\varkappa$ is a homothety with centre O and coefficients $\varkappa$, $[q] = A\,[p_c]^\alpha (\mu h)^{-1} > 0$, where $[q]$ is independent of h for $m = 1, \alpha = 2/(2\nu + 1)$.

We call the function $p_*(M_*) = F(q(M_*))$ the reduced constact pressure, and the domain $\omega_* = \{M_*: p_*(M_*) \geqslant [q]^{1/\alpha}\} \subset S_*$ is the reduced domain of elastoplastic deformation ($\omega = H_0^\varkappa(\omega_*)$).

We obtain the following system for the unknowns $q(M_*)$, S_*, h

$$\mu q(M_*) + \int\limits_{S_*} K(M_*, N_*) F(q(N_*))\, dS_{N_*} = 1 - f(M_*) \tag{1.9}$$
$$q(M_*) \geqslant 0;\quad q(L_*) = 0;\quad \mu = A\theta^{-\alpha}(\beta^\alpha h^{\alpha(m-1)-m})^{1/m}$$
$$h = h_*\beta^{1/(m+1)}(P\theta)^{m/(m+1)},\quad M_* \in S_*$$
$$\int\limits_{S_*} K(M_*, N_*) F(q(N_*))\, dS_{N_*} > 1 - f(M_*);\ q(M_*) = 0,\ M_* \notin S_*$$
$$h_* = \Big(\int\limits_{S_*} F(q(N_*))\, dS_{N_*}\Big)^{-m/(m+1)}$$

Here L_* is the boundary of the domain S_* while $q, F, S_*, \mu, \alpha, h_*$ are dimensionless quantities.

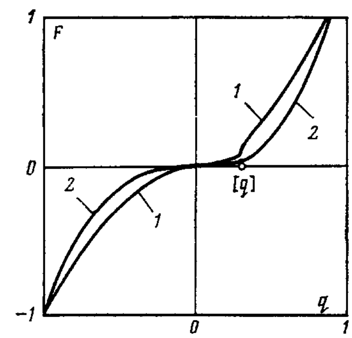

Fig.1

The scalar h_* evidently depends on $\mu, \alpha, [q], f$. The case $\mu \equiv 0$ ($[p_c] = \infty$) corresponds to the Hertz formulation of the contact problem for ideally smooth elastic bodies. For $0 < \alpha < {}^2/_3$ the function $F(q)$ is defined for all q, is continuous, increases strictly, and a constant c_* exists such that $|F(q)| \leqslant c_* |q|^{1/\alpha}$. For $0.4 < \alpha < {}^2/_3$ and $q = [q]$, the derivative $F'(q)$ is a discontinuous function ($\lim F'(q) = \infty$ as $q \to [q] + 0$). If there are no plastic strains, then the function $F(q)$ is defined for all q and is continuously differentiable for $0 < \alpha \leqslant 1$.

Fig.1 shows graphs of the function $F(q)$. Curves 1, and 2 are constructed for $[q] = 0.3$ and the values $\nu = 1.5$ and $\nu = 2.5$, respectively.

Starting from this, we distinguish two cases: a) F is a continuously differentiable function, i.e., there are no plastic deformations of the unevennesses or $0 < \alpha \leqslant 0.4$ for elastoplastic deformations of the unevenness; b) the derivative F' is a discontinuous function for $q = [q]$, i.e. elastoplastic deformations exist and $0.4 < \alpha < {}^2/_3$.

We introduce the notation $L_{\omega_*} = \{M_* : q(M_*) = [q]\}$, $S^- = S_* \setminus L_{\omega_*}$. Then the following proposition establishes the additional properties of the solution $q(M_*)$ of the system (1.9):

Proposition 3. If $f(M_*) \in \mathrm{Lip}_r(S_*)$, $0 < r < 1$, the derivative $f'(M_*)$ with respect to any direction l is continuous at the point $T \in S_*$ in case a) or at the point $T \in S^-$ in case b), then $q(M_*) \in \mathrm{Lip}_r(S_*)$ and the corresponding derivative $q'(M_*)$ is continuous for $M_* = T$ (if $T \in L_*$, then $q'(T) = \lim q'(T_1)$ as $T_1 \to T$, where T_1 is an interior point of S_* on l, and $\mathrm{Lip}_r(S_*)$ is a class of functions satisfying the Lipschitz condition with index r on S_*).

Proposition 3 is a corollary of Theorem 4 (/6/, p.422) and Theorem 1.5 (/7/, p.339). The deduction that if $f(M_*) \in \mathrm{Lip}_r(S_*)$, $0 < r < 1$ and the point $T \in L_{\omega_*}$ in case b), then the continuity of $f'(M_*)$ for $M_* = T$ does not guarantee the continuity of the corresponding derivative $q'(M_*)$ at the point $M_* = T$, follows directly from these theorems. In other words, the contact pressure $p(M)$ on the domain boundary ω will not generally by a smooth function (because of the discontinuity in the derivative $F'(q)$ for $q = [q]$ and $0.4 < \alpha < {}^2/_3$).

Examination of the contact problem for a half-space does not limit the generality. This is explained by the fact that each of the bodies making contact can be replaced by a half-space if the radii of curvature of these bodies are large compared with the size of the contact area.

2. Method of solution. System (1.9) is solved by the method of successive approximations. The crux of the method is that as $S_*^{(k)}$ approaches the domain S_* and $h^{(k)}$ approaches h $(k = 0, 1, 2, \ldots)$ a $\mu^{(k)}$ and a solution $q^{(k)}(M_*)$ of the Hammerstein integral equation (1.9) are determined by which the approximation $p_*^{(k)}(x_*, y_*)$ to the reduced contact pressure $p_*(x_*, y_*) = F(q(x_*, y_*))$ and therefore, the next approximations $S_*^{(k+1)}, h^{(k+1)}$ are determined. The construction of the sequence $S_*^{(k)}, k = 0, 1, 2, \ldots$ is here carried out as in /9/. The solution of the appropriate Hertz problem can be taken as the initial approximations $S_*^{(0)}, h^{(0)}$. Since $q(M_*) \geqslant 0$, the domain $S_*^{(0)} = \{M_*: f(M_*) \leqslant 1\} \supset S_*$ can also be taken as an initial approximation of $S_*^{(0)}$

The process of seeking the successive approximation consists of two fundamental steps: obtaining the solution $q(M_*) = q^{(k)}(M_*)$ of the integral equation (1.9) for a fixed domain of integration $S_* = S_*^{(k)}$ $(k = 0, 1, 2, \ldots)$ and construction of the next approximation $S_*^{(k+1)}$ to the domain S_* by means of the solution $q^{(k)}(M_*)$ obtained.

For large μ the integral equation (1.9) can be solved by the method of successive approximations

$$q_{i+1}(M_*) = -\frac{1}{\mu}\int_{S_*} K(M_*, N_*)F(q_i(N_*))\,dS_{N_*} + \frac{1}{\mu}(1 - f(M_*)) \tag{2.1}$$

$i = 0, 1, 2, \ldots$

Convergence of the method (2.1) was observed for $\mu > 1$ in the problems examined later. More important in practice are the values $\mu \leqslant 1$. For these values of μ and the values of the parameter α for which the operator on the left side of integral equation (1.9) is Fréchet differentiable /6/ in $C(S_*)$, Newton's method is effective

$$\begin{gathered} q_{i+1}(M_*) = q_i(M_*) + \Delta_i(M_*);\ i = 0, 1, 2, \ldots \\ \mu\Delta_i(M_*) + \int_{S_*} K(M_*, N_*)F_q'(q_i(N_*))\,\Delta_i(N_*)\,dS_{N_*} = \\ 1 - f(M_*) - \varepsilon_i(M_*) \\ \varepsilon_i(M_*) = \mu q_i(M_*) + \int_{S_*} K(M_*, N_*)F(q_i(N_*))\,dS_{N_*} \end{gathered} \tag{2.2}$$

In case of the non-differentiability of the operator mentioned (for $0.4 < \alpha < {}^2/_3$ and elastoplastic unevenness deformations), an iteration process is used

$$\begin{gathered} q_{i+1}(M_*) = q_i(M_*) + \gamma\Delta_i(M_*);\ 0 < \gamma \leqslant 1,\ i = 0, 1, 2, \ldots \\ \mu\Delta_i(M_*) + \int_{S_*} K(M_*, N_*)\,n\,|q_i(N_*)|^{n-1}\Delta_i(N_*)\,dS_{N_*} = 1 - f(M_*) - \varepsilon_i(M_*) \end{gathered} \tag{2.3}$$

where the quantity ε_i is defined in (2.2) and $n = 1/\alpha$.

We now consider the process of obtaining successive approximations of $S_*^{(k)}, k = 0, 1, 2, \ldots$. We go over to polar coordinates r, φ. Let $r = \rho_k(\varphi)$ be the equation of the boundary of the domain $S_*^{(k)}$, and $q^{(k)}(r, \varphi)$ the solution of the integral equation (1.9) corresponding to this boundary. We introduce the auxiliary function $\rho_k^+(\varphi)$ by the following method:

$$|p_*^{(k)}(\rho_k^+, \varphi)| = \min_{0 \leqslant r \leqslant \rho_k} |p_*^{(k)}(r, \varphi)| \equiv \min_{0 \leqslant r \leqslant \rho_k} |F(q^{(k)}(r, \varphi))| \tag{2.4}$$

If $\rho_k^+ = \rho_k$ and $p_*^{(k)}(\rho_k, \varphi) \neq 0$, then we set

$$\rho_k^+ = \rho_k + \left(\frac{\partial}{\partial r} p_*^{(k)}(\rho_k, \varphi)\right)^{-1} p_*^{(k)}(\rho_k, \varphi)$$

and $p_*^{(k)}(\rho_k^+, \varphi) = 0$.

Then the iteration process that was used to seek the domain S_* is written in the form

$$\begin{gathered} \rho_{k+1} = \rho_k + \gamma(\rho_k - \rho_k^+)\frac{p_*^{(k)}(\rho_k, \varphi)}{|p_*^{(k)}(\rho_k^+, \varphi)| + |p_*^{(k)}(\rho_k, \varphi)|} \\ \rho_0 = (f(\cos\varphi, \sin\varphi))^{-1/m};\ k = 0, 1, 2, \ldots;\ 0 < \gamma \leqslant 1 \end{gathered} \tag{2.5}$$

The iteration method (2.5) can be considered as a set of one-dimensional iteration processes operating on each ray $\varphi = \text{const}$ in the plane Ox_*y_* and leaving from the point O. For axisymmetric problems the method of bisection is also effective in looking for the contact circle.

3. Numerical analysis. The examples examined below enable us to estimate the possibilities of the method and the influence of the roughness. All the calculations were performed on the BESM-6 computer. Systems of linear algebraic equations corresponding to discrete formulations of the problems were solved by Gauss's method. The confidence in the results obtained was determined (mainly) by their stability to an increase in the number of mesh nodes of double or more.

Example 1. A pyramid $z = -\beta \max(|x|, |y|)$, $\beta = \operatorname{ctg}\gamma$ (γ is the angle between the z axis and the pyramid face) is impressed into the half-space $z \geqslant 0$. The contact pressure distribution $p_*(M_*)$ obtained for elastic deformation of the unevenness is shown in Fig.2 for $\mu = 0.5$ and $\alpha = 0.5$. We also show here for comparison (the dashes) the boundary of the contact area corresponding to Hertz's formulation of the problem, $\mu \equiv 0$ /9/. The value $h_* = 2.35$ is obtained instead of 2.03 in Hertz's formulation.

The contact pressure distribution on the Ox_* axis are shown with and without taking account (dashed curve) of the rough layer in Fig.3.

The contact area S_* and the contact pressure distribution $p_*(M_*)$ on it ($h_* = 2.81$) are shown in Fig.4 for $\mu = 0.5$, $\alpha = 0.2$, $|q| = 0.4$. The dashed line in this figure bounds the reduced domain of elastoplastic deformations of the unevenness $\omega_* = \{M_* : p_*(M_*) \geqslant 0.4^5\}$.

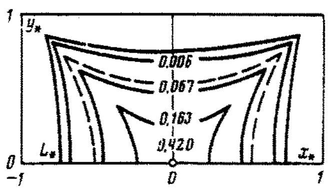

Fig.2

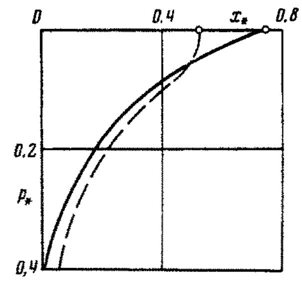

Fig.3

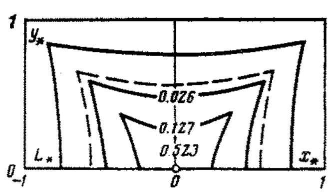

Fig.4

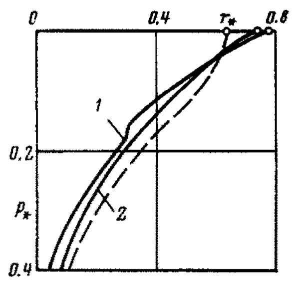

Fig.5

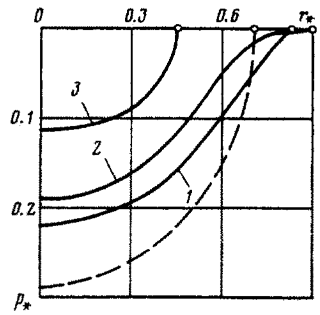

Fig.6

Despite the fact that the stamp is not smooth, the contact pressure when there is roughness is determined by the continuous function $p_*(M_*)$ unlike Hertz's formulation of this problem (see Appendix 1).

Equations (2.1)—(2.5 were discretized as in /9/. The symmetry of the solutions $q(M_*)$, S_* relative to the four axes was taken into account in compiling the system of linear algebraic equations. The order of the system reached 66. The process (2.5) ($\gamma = 0{,}5$) was halted when $|p_*^{(k)}(\rho_k(\varphi), \varphi)| \leqslant 5\cdot 10^{-5}$. The computation time was 8 min for no plastic strains, and 15 min when there were plastic strains.

Example 2. A cone $z = -\beta\sqrt{x^2 + y^2}$, $\beta = \operatorname{ctg}\gamma$ (γ is the angle between the cone generatrix and the z axis) is impressed in a half-space $z \geqslant 0$. The values of the function $h_*(\mu, \alpha)$ obtained and the reduced radius of the contact area $a_*(\mu, \alpha)$ are given for elastic deformations of the unevenness on the left side of the table (the top row is h_* and the bottom is a_*).

Table

μ	$\alpha = 0.2$	0,6	1,0	$\alpha = 0.2$	0.6	1.0
0	2.22 0.65	2.22 0.65	2.22 0.65	2.23 0.72	2.23 0,72	2,23 0.72
0.2	2.51 0.78	2.32 0.71	2.26 0.68	2.53 0.82	2.35 0.77	2.28 0.74
0.6	3.38 0.90	2.55 0,79	2.35 0.72	3.48 0.90	2.63 0.83	2.41 0,78
1.0	4.90 0.95	2.79 0.83	2.44 0.75	5.13 0.95	2.94 0,86	2.52 0,80

Distributions of the contact pressures $p_*(r_*)$, $r_* = \sqrt{x_*^2 + y_*^2}$ are displayed in Fig.5 for different values of the dimensionless parameters. The dashes show the pressure distribution corresponding to Hertz's formulation of the problem, $\mu \equiv 0$ /3/. Line *1* corresponds to the values $\mu = 0.4, \alpha = 0.5, [q] = 0.4$ for elastoplastic deformations of the unevenness in the domain $\omega_* = \{r_* : p_*(r_*) \geqslant 0.4^2\}$; line *2* corresponds to the values $\mu = 0.2$, $\alpha = 0.4$, $[q] = 0.05$ for elastoplastic deformations of the unevenness in the domain $\omega_* = \{r_* : p_*(r_*) \geqslant 0.05^{2.5}\}$.

When there is roughness the contact pressure $p_*(r_*)$ is a continuous function (Proposition 1) but not generally smooth, as mentioned above. The calculations showed a high probability of a break on the boundary of the domain ω_* of the graph of the function $p_*(r_*)$ for $\mu = 0.4$, $\alpha = 0.5, [q] = 0.4$ (Fig.5, curve *1*).

Discretization was realized taking axial symmetry into account. The contact circle $S_*^{(k)}$ of radius $a_*^{(k)}$ was divided into area elements of equivalent area. The partition along the radius was assumed to be uniform. The mesh nodes were placed within the area elements. The polar angles of the nodes located on the circle $r = a_*^{(k)}(j-1)/(N-1)$ $(j = 1, 2, \ldots, N)$, equal

$$\varphi_i = \begin{cases} \dfrac{\pi(i-1)}{4(j-1)} \text{ for } i = 1,2,\ldots,8(j-1) \text{ and } j = 2,3,\ldots,(N-1) \\ \dfrac{\pi(i-1)}{2(N-1)} \text{ for } i = 1,2,\ldots,4(N-1),\ j = N \end{cases}$$

Here N is the order of the system of linear algebraic equations. Results are given for $N = 11$ and 401 nodes. The iteration process was terminated when the following condition was satisfied: $|p_*^{(k)}(a_*^{(k)})| \leqslant 5\cdot10^{-8}$. The computation time for one value of the group of parameters $\mu, \alpha, [q]$ is 30 seconds for elastic deformations, 1 minute for elastoplastic deformations and a smooth function F, and 15 minutes for elastoplastic deformations and a discontinuous function F'.

Example 3. A paraboloid $z = -\beta(x^2 + y^2)$, $\beta = (2R)^{-1}$ (R is the radius of curvature of the paraboloid at the vertex) is impressed into a half-space $z \geqslant 0$. The calculated values of the functions $h_*(\mu, \alpha)$ and the radius of the contact area $a_*(\mu, \alpha)$ are presented on the right side of the table for elastic deformations of the unevenness (upper row for h_* and lower for a_*). The distributions of the contact pressure $p_*(r_*)$, $r_* = \sqrt{x_*^2 + y_*^2}$ are shown in Fig.6. The solution of Hertz's problem $(\mu \equiv 0)$ is shown by dashes. Curve *1* corresponds to the values $\mu = 0.6$, $\alpha = 0.6$, curve *2* to the values $\mu = 0.6$, $\alpha = 0.2$, and curve *3* to the values $\mu = 0.6$, $\alpha = 0$.

For $\alpha = 0$ the roughness layer does not hinder insertion of the stamp and the parameter μ indicates the thickness of the rough layer with respect to h. Hence, for small α (when the resistance of the rough layer to insertion of the stamp is not large) and high μ (when in addition to the low resistance of the rough layer, its thickness relative to h is considerable), the contact pressure in a fairly broad neighbourhood of the contact area boundary is almost zero, and similar to the Hertz value near the centre of the contact area (Fig.6, curve *2*). In this case the main resistance to stamp insertion turns out to be the elastic half-space: the stamp seems to press through the weak rough layer and "be seated" on the more rigid elastic foundation. The presence of the point of inflection on curve *2* is explained thereby.

Results are give for the same discretization of the problem as in Example 2. The computation time for one value of the pair of parameters μ, α is 30 seconds. An analogous problem was solved by another method in /10/. The numerical results presented there for graphs 1 and 2 (the radii of the contact areas, and the contact pressures) are practically in agreement with the results of the present paper (the values of the parameters $\mu = 0$ and $\mu = 0.5, \alpha = {}^2/_3$) correspond, respectively, to graphs 1 and 2 in /10/). The first initial approximation in methods (2.1)–(2.3) was assumed to be equal, $q_0(M_*) = 0.5$, in all the examples.

The influence of body roughness on the contact mechanical characteristics can be determined on the basis of the analysis performed. As mentioned, the case $\mu \equiv 0$ corresponds to Hertz' formulation of the problem for ideally smooth bodies. The higher the value of the μ, the greater the difference between problem (1.9) and the problem in Hertz'f formulation. It follows from relations (1.9) that for small depths of insertion h (i.e., for fairly small values of P), the value of the parameter μ can be large. Large values of μ can be achieved for comparatively good treatment of the surface for high-modulus materials and for surfaces with low treatment quality, i.e. for large A and small α. Body roughness results mainly in smoothing of the contact pressure diagrams, and an increase in the convergence between bodies as well as in the contact area (as compared with the corresponding contact characteristics of ideally smooth bodies).

REFERENCES

1. DEMKIN N.B., Contact Between Rough Surfaces. Nauka, Moscow, 1970.
2. KRAGEL'SKII I.V., DOBYCHIN M.N. and KOMBALOV V.S., Friction and Wear Computational Principles. Mashinostroenie, Moscow, 1977.

3. GALIN L.A., Contact Problems of Elasticity and Viscoelasticity Theories. Nauka, Moscow, 1980.
4. GORYACHEVA I.G., Plane and axisymmetric contact problems for rough elastic bodies, PMM, Vol.43, No.1, 1979.
5. GALIN L.A., ed., Development of the Theory of Contact Problems in the USSR. Nauka, Moscow, 1976.
6. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis, Nauka, Moscow, 1977.
7. ZABREIKO P.P., KOSHELEV A.I., KRASNOSEL'SKII M.A., MIKHLIN S.G., RAKOVSHCHIK L.S. and STETSENKO V.YA., Integral Equations, Nauka, Moscow, 1968.
8. VAINBERG M.M., The Variational Method and the Method of Montonic Operators in the Theory of Non-linear Equations. Nauka, Moscow, 1972.
9. GALANOV B.A., On an approximate solution of certain problems of the elastic contact of two bodies, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.5, 1981.
10. RABINOVICH A.S., Axisymmetric contact problem for rough elastic bodies. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela. No.4, 1975.

Translated by M.D.F.

PMM U.S.S.R.,Vol.48,No.6,pp.757-764,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

EQUILIBRIUM OF AN ELASTIC LAYER WEAKENED BY PLANE CRACKS*

B.I. SMETANIN and B.V. SOBOL'

The spatial problem of the elastic equilibrium of a layer in whose middle plane there is a system of cracks is considered. The cracks are maintained open under the action of a normal load applied to their edges. The layer faces are compressed between two rigid smooth foundations. The problem is reduced to solving an integral equation of the first kind. The asymptotic methods of "large and small λ" /1/ as well as the method of successive approximations and a variational method are used to construct the solutions of this equation for elliptically and rectangularly shaped cracks in different ranges of variation of the geomtrical parameters.

1. Formulation of the problem, properties of the kernel of the integral equation. Let a domain occupied by an elastic medium be determined by the inequalities $|z| \leqslant h$, $|x| < \infty$, $|y| < \infty$. A crack occupying a certain domain Ω in planform is in the $z = 0$ plane. A load $\sigma_z = -p(x, y)$, $z = \pm 0$ is applied to the crack edges. The following conditions are realized on the faces of the layer, at $z = \pm h$: $W = 0$, $\tau_{xz} = \tau_{yz} = 0$, where W is the projection of the displacement vector on the Oz axis, and τ_{xz}, τ_{yz} are the stress tensor components.

The problem under consideration is reduced to the solution of an integral equation of the first kind by the methods of integral transformation:

$$-\Delta \iint_{\Omega} \gamma(\xi, \eta) \frac{d\xi\, d\eta}{R} + \iint_{\Omega} \gamma(\xi, \eta) K_1\left(\frac{R}{h}\right) d\xi\, d\eta = \frac{2\pi p(x, y)}{\Theta} \tag{1.1}$$

$$(x, y) \in \Omega$$

$$K_1(\alpha) = \frac{1}{h^3} \int_0^\infty [L(u) - 1]\, u^2 J_0(\alpha u)\, du, \quad L(u) = \frac{\operatorname{sh} 2u + 2u}{\operatorname{ch} 2u - 1}$$

$$\gamma(x, y) = W(x, y, 0), \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$R = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \quad \Theta = \frac{E}{2(1-\nu^2)}$$

where E is Young's modulus, ν is Poisson's ratio, and $J_0(x)$ is the Bessel function of the first kind.

As a result of utilizing the well-known integral representations /1/, the integral equation (1.1) can be converted to the form

*Prikl.Matem.Mekhan.,48,6,1030-1038,1984